Electromagnetism: AC and RLC Circuits

FIZIKA SPhO Training

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1 Notes

We shall first expand on our knowledge of circuits past resistors and introduce the ideas of capacitors and inductors and their usages in circuits. Then, we shall examine circuits with sinusoidal time dependence and introduce the concept of phasors, which can help simplify these circuits greatly.

1.1 Circuit Elements

The main circuit elements that you will encounter are **resistors**, **capacitors**, **inductors**, **voltage sources**, **current sources** and **switches**. There are other circuit elements that you might encounter in more advanced settings, such as transformers or diodes, but we shall not study them in detail.

When these elements are connected in a circuit, the voltage and current difference across the elements are determined by differential equations, which are to be solved to obtain the full time dependence.

1.1.1 Resistors, Capacitors and Inductors

As we have learnt, the equations that govern the relationship between the voltages across and currents through resistors, capacitors and inductors are as follows.

$$V = IR \tag{1}$$

$$I = C\frac{\mathrm{d}V}{\mathrm{d}t} \tag{2}$$

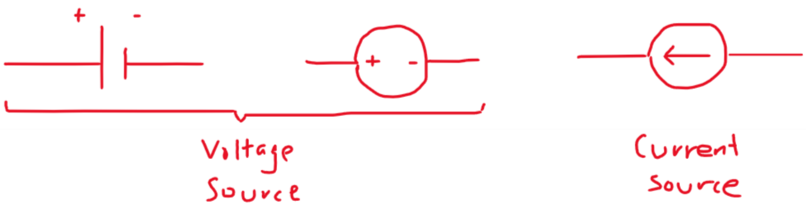
$$V = L\frac{\mathrm{d}I}{\mathrm{d}t} \tag{3}$$

In the equations above, the sign is taken to be positive if we take the current flowing through circuit element to be in the direction of the decrease in voltage. Likewise, the signs in the equations above should be negative if we taken the current flowing through the circuit to be in the direction of increase in voltage. This is termed the *passive sign convention*. It is not necessary to adopt this convention, but it is highly recommended as it heavily simplifies keeping track of the signs.

Circuits involving these components are called **RLC circuits**.

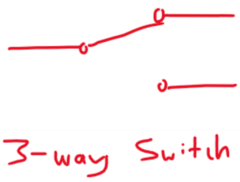
1.1.2 Voltage and Current Sources

Voltage sources maintain a voltage difference V across itself, while current sources maintain a current I across itself. You will see voltage sources more than current sources – batteries are an example of a voltage source.



1.1.3 Switches

A switch is a circuit element that can be switched on or off to close or open a wire. When the switch is toggled on, the switch can be regarded as a wire of zero resistance that freely allows current to flow through. When the switch is toggled off, the switch can be regarded as a missing connection with infinite resistance that does not allow current to flow through.



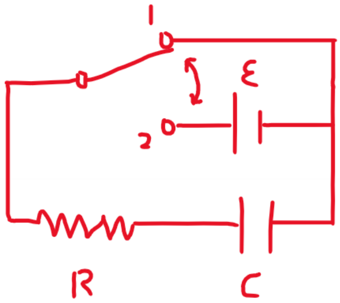
1.2 First-order Circuits

We're already studied zeroth-order circuits – these are circuits generally only involving resistors, which do not exhibit any time dependence.

Now, let us first study first-order circuits – these are circuits that generally do not involve both a capacitor and an inductor. These circuits are usually very simple to deal with, and the solutions to the differential equations that govern their time dependence are simple enough to be memorised.

1.2.1 A Basic RC Circuit

To consider a basic example, let's consider a circuit containing a voltage source ε and a switch with a resistor R and a capacitor C in series.



The switch starts at position 1 and the capacitor is initially uncharged. At $t = 0$, the switch flipped to position 2, and we want to solve for the voltage across the capacitor V_{C} and the current I as functions of time for $t \geq 0$.

To do so, the capacitor governs that

$$I = C\frac{\mathrm{d}V_{\mathrm{C}}}{\mathrm{d}t}$$

so applying Kirchhoff's voltage law on the loop gives

$$\varepsilon - IR - V_{\mathrm{C}} = 0 \quad \Longrightarrow \quad RC\frac{\mathrm{d}V_{\mathrm{C}}}{\mathrm{d}t} + V_{\mathrm{C}} = \varepsilon$$

This is a first-order linear differential equation which can be solved easily to get

$$V_{\mathrm{C}}(t) = \varepsilon\left(1 - e^{-\frac{t}{RC}}\right)$$

where we used the initial condition that the capacitor is initially uncharged, i.e. $V_{\mathrm{C}}(0) = 0$. Hence, we can also calculate the current as

$$I(t) = \frac{\varepsilon}{R}e^{-\frac{t}{RC}}$$

This is the *charging* RC circuit response.

Now let's suppose that the switch is at position 2 for a long time, so the capacitor has voltage ε. At $t = 0$ (redefining t), the switch is flipped to position 1, and we want to solve for the voltage across the capacitor V_{C} and the current I as functions of time for $t \geq 0$.

To do so, we can proceed by a similar method to obtain the differential equation

$$RC\frac{\mathrm{d}V_{\mathrm{C}}}{\mathrm{d}t} + V_{\mathrm{C}} = 0$$

which can be solved to get

$$V_{\mathrm{C}}(t) = \varepsilon e^{-\frac{t}{RC}}$$

as well as

$$I(t) = -\frac{\varepsilon}{R}e^{-\frac{t}{RC}}$$

This is the *discharging* RC circuit response.

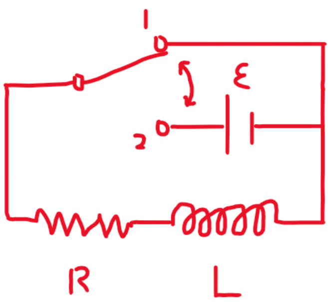
Note one very important characteristic of the charging and discharging circuit responses: their time dependence is *exponential*. We can then define the **time constant** of the circuit to be

$$\tau = RC \tag{4}$$

which dictates the exponential rate of the time dependence of the circuit voltages and currents.

1.2.2 A Basic RL Circuit

RL circuits behave similarly to RC circuits. To consider another basic example, let's consider a circuit containing a voltage source ε and a switch with a resistor R and an inductor L in series.



The switch starts at position 1 and the inductor is initially unenergised. At $t = 0$, the switch is flipped to position 2, and we want to solve for the voltage across the inductor V_{L} and the current I as functions of time for $t \geq 0$.

To do so, the inductor governs that

$$V_{\text{L}} = L\frac{\mathrm{d}I}{\mathrm{d}t}$$

so applying Kirchhoff's voltage law on the loop gives

$$\varepsilon - IR - V_{\text{L}} = 0 \quad \implies \quad \frac{L}{R}\frac{\mathrm{d}I}{\mathrm{d}t} + I = \frac{\varepsilon}{R}$$

This is a first-order linear differential equation which can be solved easily to get

$$I(t) = \frac{\varepsilon}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

where we used the initial condition that the inductor is initially unenergised, i.e. $I(0) = 0$. Hence, we can also calculate the voltage as

$$V_{\text{L}}(t) = \varepsilon e^{-\frac{Rt}{L}}$$

This is the *energising* RL circuit response.

Now let's suppose that the switch is at position 2 for a long time, so the inductor has current $\frac{\varepsilon}{R}$. At $t = 0$ (redefining t), the switch is flipped to position 1, and we want to solve for the voltage across the inductor V_{L} and the current I as functions of time for $t \geq 0$.

To do so, we can proceed by a similar method to obtain the differential equation

$$\frac{L}{R}\frac{\mathrm{d}I}{\mathrm{d}t} + I = 0$$

which can be solved to get

$$I(t) = \frac{\varepsilon}{R}e^{-\frac{Rt}{L}}$$

as well as

$$V_{\text{L}}(t) = -\varepsilon e^{-\frac{Rt}{L}}$$

This is the *deenergising* RL circuit response.

Once again, the energising and deenergising circuit responses are exponential, and we can define the time constant of the circuit to be

$$\tau = \frac{L}{R} \tag{5}$$

1.2.3 General Circuits with One Capacitor or Inductor

The two examples we've shown previously are very basic. What if we had any number of voltage sources, current sources or resistors instead? Using Kirchhoff's laws, nodal analysis or mesh analysis is going to be very tedious.

It turns out that if the circuit only has one capacitor or inductor, solving for the circuit response is actually very easy. In general, if we have an RC or RL circuit that consists of a single capacitor or inductor, the following corollary is true:

Corollary. In a circuit containing only voltage sources, current sources, resistors and a single capacitor C or inductor L,

1. the time dependence of the voltages and currents in the circuit are a linear function of an exponential in time, and

2. the time constant of the circuit will be

$$\tau = R_{\text{eff}}C \quad \text{or} \quad \tau = \frac{L}{R_{\text{eff}}} \tag{6}$$

where R_{eff} is the effective resistance across the terminals of the capacitor or inductor respectively.

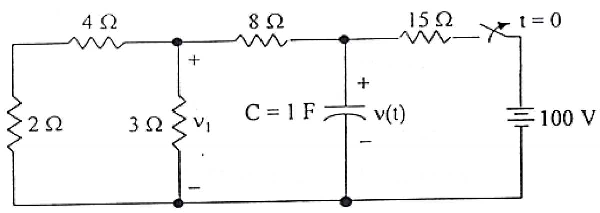
We shall omit a formal proof, but it is in fact easy to see that this corollary comes about from Thévenin's theorem: we can just replace everything across the two terminals of the capacitor or inductor with their Thévenin equivalent circuit, and then the analysis of the circuit response proceeds as in the previous subsections. How convenient!

This can massively simplify the process of solving for the time dependence of a parameter in an RC or RL circuit with a single capacitor or inductor, since all we need to do is

1. calculate the time constant of the circuit,
2. determine the initial and limiting final values of the parameter, and
3. find a linear function of an exponential in time with the calculated time constant that has the corresponding initial value and limiting final value.

We shall illustrate this process with a few examples.

Example 1.1. (Ricardo) The switch was closed for a long time before $t = 0$. Determine the capacitor charge $v(t)$ for $t > 0$.



First, the effective resistance between the terminals of the capacitor after the switch is opened is

$$R_{\text{eff}} = 8 + \left(\frac{1}{3} + \frac{1}{2+4}\right)^{-1} = 10\ \Omega$$

so the time constant of the circuit is

$$\tau = R_{\text{eff}}C = 10\ \text{s}$$

Initially, since the switch was closed for a long time, the capacitor has reached equilibrium, so there is no current flowing through the capacitor. Hence, we just need to calculate the potential across the terminals of the capacitor where the capacitor can be treated as open, so by the potential divider rule,

$$v_0 = \frac{8 + \left(\frac{1}{3} + \frac{1}{4+2}\right)^{-1}}{15 + \left(8 + \left(\frac{1}{3} + \frac{1}{4+2}\right)^{-1}\right)} \cdot 100 = 40\ \text{V}$$

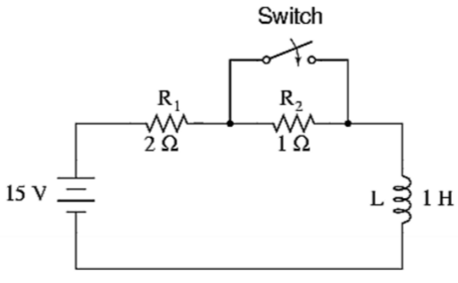
After a very long time, since the switch is opened, no current flows through the voltage source, so the capacitor must discharge and

$$v_\infty = 0\ \text{V}$$

Therefore, we must have

$$v(t) = v_\infty + (v_0 - v_\infty)e^{-\frac{t}{\tau}} = 40e^{-\frac{t}{10\ \text{s}}}\ \text{V}.$$

Example 1.2. (Ricardo) The switch was open for a long time before $t = 0$. Determine the inductor current $i(t)$ for $t > 0$.



First the effective resistance between the terminals of the inductor after the switch is closed is

$$R_{\text{eff}} = 2\ \Omega$$

so the time constant of the circuit is

$$\tau = \frac{L}{R_{\text{eff}}} = 0.5\ \text{s}$$

Initially, since the switch was open for a long time, the inductor has reached equilibrium, so there is no voltage difference across the inductor. Hence, we just need to calculate the current through the inductor where the inductor can be treated as closed, so we have

$$i_0 = \frac{15}{2+1} = 5\ \text{A}$$

After a very long time, since the switch is closed, no current flows through the 1 Ω resistor, so the current changes and

$$i_\infty = \frac{15}{2} = 7.5\ \text{A}$$

Therefore, we must have

$$i(t) = i_\infty + (i_0 - i_\infty)e^{-\frac{t}{\tau}} = 2.5\left(3 - e^{-\frac{t}{0.5\ \text{s}}}\right)\ \text{A}.$$

1.2.4 Short and Long Term Behaviour

Sometimes, it would be useful to analyse the short and long term behaviour of circuits, for example when using the methods above. Let's look at the equations that govern the voltage and current of a capacitor and inductor again,

$$I = C\frac{\text{d}V}{\text{d}t} \qquad V = L\frac{\text{d}I}{\text{d}t}$$

These equations must imply that:

1. **The voltage across a capacitor is continuous with time**, as a discontinuous voltage would result in infinite current.
2. **The current across an inductor is continuous with time**, as a discontinuous current would result in infinite voltage.

On the other hand, resistor voltages and currents can be discontinuous as they are directly proportional. This allows us to deduce the short and long term behaviour of these circuit elements after a sudden change (for example, a switch being flipped) using the initial and equilibrium conditions respectively.

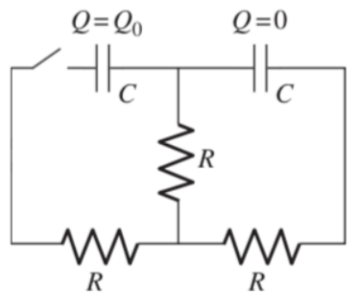
1. For a capacitor, its voltage right after a sudden change is the same as its voltage right before the change, and its current after a long time is zero.
2. For an inductor, its current right after a sudden change is the same as its current right before the change, and its voltage after a long time is zero.

This will be useful when solving circuits involving these circuit elements.

1.2.5 General First-order Circuits

For circuits containing only one capacitor or inductor, the method shown above will suffice. However, for circuits with more than one capacitor or inductor, this method may no longer apply, and we unfortunately have to solve the associated differential equations.

Example 1.3. (Purcell) Consider the circuit below. The circuit contains two identical capacitors and three identical resistors, with initial charges shown above. If the switch is closed at $t = 0$, find the maximum charge that the right capacitor achieves.



Let the two mesh currents be i_1, i_2. Applying mesh analysis gives

$$V_1 - 2i_1R + i_2R = 0 \qquad V_2 + i_1R - 2i_2R = 0$$

which can be expressed in terms of the charges q_1, q_2 as

$$\frac{q_1}{C} + 2R\frac{\mathrm{d}q_1}{\mathrm{d}t} - R\frac{\mathrm{d}q_2}{\mathrm{d}t} = 0 \qquad \frac{q_2}{C} - R\frac{\mathrm{d}q_1}{\mathrm{d}t} + 2R\frac{\mathrm{d}q_2}{\mathrm{d}t} = 0$$

This is a set of coupled differential equations, which may seem intimidating to solve, but can actually just be accomplished by adding and subtracting the two equations to get

$$\frac{q_1 + q_2}{C} + R\frac{\mathrm{d}}{\mathrm{d}t}(q_1 + q_2) = 0 \qquad \frac{q_1 - q_2}{C} + 3R\frac{\mathrm{d}}{\mathrm{d}t}(q_1 - q_2) = 0$$

which is each a first-order linear differential equation. Solving gives

$$q_1 + q_2 = Q_0 e^{-\frac{t}{RC}} \qquad q_1 - q_2 = Q_0 e^{-\frac{t}{3RC}}$$

using the initial conditions, which can be solved for the individual charges

$$q_1 = \frac{Q_0}{2}\left(e^{-\frac{t}{RC}} + e^{-\frac{t}{3RC}}\right) \qquad q_2 = \frac{Q_0}{2}\left(e^{-\frac{t}{RC}} - e^{-\frac{t}{3RC}}\right)$$

To find the maximum charge on the right capacitor, we can set the derivative to be zero to get

$$\frac{Q_0}{2}\left(-\frac{1}{RC}e^{-\frac{t}{RC}} + \frac{1}{3RC}e^{-\frac{t}{3RC}}\right) = 0 \quad \Longrightarrow \quad t = \left(\frac{3}{2}\ln 3\right)RC$$

which can be substituted to find the maximum charge as

$$q_{2,\max} = \frac{Q_0}{3\sqrt{3}}.$$

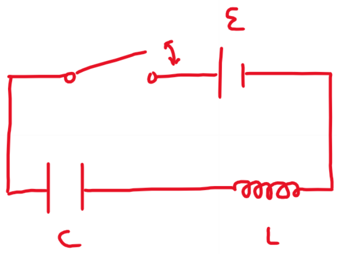
These questions are generally more advanced, but solving them is a good skill to have.

1.3 Second-order Circuits

Let us now study second-order circuits – these are circuits that generally involve both a capacitor and an inductor. These circuits are generally more complicated to solve, and require an understanding of damping to qualitatively deduce their behaviour.

1.3.1 A Basic LC Circuit

We shall first omit resistors and consider a circuit containing only a switch with both a capacitor C and an inductor L in series.



The switch starts open and the capacitor was charged by a voltage source so that it has initial voltage ε. At $t = 0$, the switch is closed, and we want to solve for the voltage across the capacitor V_{C}, the voltage across the inductor V_{L} and the current I as functions of time for $t \geq 0$.

To do so, the capacitor and inductor govern that

$$I = C\frac{\mathrm{d}V_{\mathrm{C}}}{\mathrm{d}t} \qquad V_{\mathrm{L}} = L\frac{\mathrm{d}I}{\mathrm{d}t}$$

so applying Kirchhoff's voltage law on the loop gives

$$V_{\mathrm{C}} + V_{\mathrm{L}} = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}^2 V_{\mathrm{C}}}{\mathrm{d}t^2} + \frac{V_{\mathrm{C}}}{LC} = 0$$

This is a second-order linear differential equation. In fact, this is the equation for simple harmonic motion, so the current oscillates sinusoidally as

$$V_{\mathrm{C}} = V_0 \cos\left(\frac{t}{\sqrt{LC}} + \varphi\right)$$

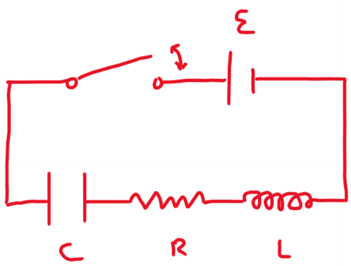
for some I_0, φ, which can be used to solve for the required quantities with the initial conditions gives

$$V_{\mathrm{C}} = \varepsilon \cos\frac{t}{\sqrt{LC}} \qquad V_{\mathrm{L}} = -\varepsilon \cos\frac{t}{\sqrt{LC}} \qquad I = -\varepsilon\sqrt{\frac{C}{L}} \sin\frac{t}{\sqrt{LC}} \tag{7}$$

Therefore, the voltages on each circuit element and the current through the loop will oscillate forever without dissipating energy.

1.3.2 Basic RLC Circuits

Let's now add a resistor R in series into the circuit above.



Once again, we want to solve for the voltage across the capacitor V_C, the voltage across the inductor V_L, the voltage across the resistor V_R and the current I as functions of time for $t \geq 0$. Similarly, we have that

$$I = C\frac{\mathrm{d}V_\text{C}}{\mathrm{d}t} \qquad V_\text{L} = L\frac{\mathrm{d}I}{\mathrm{d}t} \qquad V_\text{R} = IR$$

so applying Kirchhoff's voltage law on the loop gives

$$V_\text{C} + V_\text{L} + V_\text{R} = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}^2V_\text{C}}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}V_\text{C}}{\mathrm{d}t} + \frac{V_\text{C}}{LC} = 0$$

This is also a second-order linear differential equation. This is the equation for a damped oscillator! Therefore, we can use our knowledge of oscillations to determine the circuit response. Here the damping coefficient and (undamped) resonant frequency of the circuit is

$$\gamma = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \tag{8}$$

We can also define another parameter called the *quality factor*,

$$Q = \frac{\omega_0}{2\gamma} = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{9}$$

and we get three cases for the circuit response:

1. If $\gamma < \omega_0$, i.e. $Q > \frac{1}{2}$, then the circuit response is **underdamped** and takes the form

$$f(t) = e^{-\gamma t}\left(A_1 \cos\omega t + A_2 \sin\omega t\right) \tag{10}$$

   for some A_1, A_2, where $\omega = \sqrt{\omega_0^2 - \gamma^2}$.

2. If $\gamma = \omega_0$, i.e. $Q = \frac{1}{2}$, then the circuit response is **critically damped** and takes the form

$$f(t) = (A_1 + A_2 t)e^{-\gamma t} \tag{11}$$

   for some A_1, A_2.

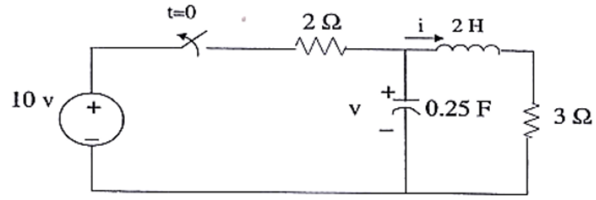
3. If $\gamma > \omega_0$, i.e. $Q < \frac{1}{2}$, then the circuit response is **overdamped** and takes the form

$$f(t) = A_1 e^{-s_1 t} + A_2 e^{-s_2 t} \tag{12}$$

   for some A_1, A_2, where $s_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$.

From there, the circuit response can then be solved for using the initial conditions. Let's give one example.

Example 1.4. (Ricardo) For the following circuit, given that the switch was closed for a long time before $t = 0$, find the expressions for $i(t)$ and $v(t)$ at $t > 0$.



When the switch is opened at $t = 0$, only the loop on the right is active, and so the circuit behaves as an RLC circuit. The damping coefficient and (undamped) resonant frequency are

$$\gamma = \frac{R}{2L} = 0.7500\ \mathrm{s}^{-1} \qquad \omega_0 = \frac{1}{\sqrt{LC}} = 1.414\ \mathrm{s}^{-1}$$

so $\gamma < \omega_0$. Hence, the circuit is underdamped. We have

$$\omega = \sqrt{\omega_0^2 - \gamma^2} = 1.198\ \mathrm{s}^{-1}$$

and so

$$v(t) = e^{-\gamma t}\left(v_1 \cos\omega t + v_2 \sin\omega t\right)$$

for some v_1, v_2. We can also take the time derivative to get

$$\frac{\mathrm{d}v}{\mathrm{d}t}(t) = e^{-\gamma t}\left((-\gamma v_1 + \omega v_2)\cos\omega t + (-\omega v_1 - \gamma v_2)\sin\omega t\right)$$

Now, we analyse the initial conditions. The switch was closed for a long time before $t = 0$. At steady state, the capacitor behaves as if it were open and the inductor behaves as if it was closed, so we just have both resistors in series with the voltage source, giving

$$v(0) = \frac{3}{2+3}\cdot 10 = 6\ \mathrm{V} \qquad i(0) = \frac{10}{2+3} = 2\ \mathrm{A}$$

The second condition, using the equation that governs the capacitor, gives

$$i = -C\frac{\mathrm{d}v}{\mathrm{d}t} \qquad \Longrightarrow \qquad \frac{\mathrm{d}v}{\mathrm{d}t}(0) = -\frac{i(0)}{C} = -8\ \mathrm{V/s}$$

Substituting the initial conditions and solving the simultaneous equations, we get

$$v_1 = 6.000\ \mathrm{V} \qquad v_2 = -2.921\ \mathrm{V}$$

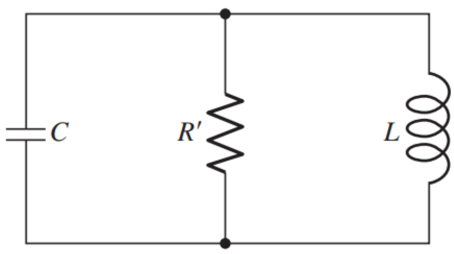
which finally gives us

$$v(t) = e^{-\left(0.7500\ \mathrm{s}^{-1}\right)t}\left(6.000\cos\left(\left(1.198\ \mathrm{s}^{-1}\right)t\right) - 2.921\sin\left(\left(1.198\ \mathrm{s}^{-1}\right)t\right)\right)\ \mathrm{V}$$
$$i(t) = -C\frac{\mathrm{d}v}{\mathrm{d}t}(t) = e^{-\left(0.7500\ \mathrm{s}^{-1}\right)t}\left(-2.000\cos\left(\left(1.198\ \mathrm{s}^{-1}\right)t\right) - 1.249\sin\left(\left(1.198\ \mathrm{s}^{-1}\right)t\right)\right)\ \mathrm{A}.$$

1.3.3 General RLC Circuits

For circuits where the resistor, capacitor and inductor are no longer in series, we have to, again, solve the associated differential equations.

Example 1.5. (Purcell) In the resonant circuit in the figure below, the dissipative element is a resistor R' connected in parallel, rather than in series, with the LC combination. Work out the differential equation for V that applies to this circuit. Find also the conditions on the solution analogous to those that hold in the series RLC circuit. If a series RLC and a parallel R'LC circuit have the same L, C, and Q (quality factor, not charge), how must R be related to R'?



We have

$$I_C = C\frac{\mathrm{d}V}{\mathrm{d}t} \qquad V = L\frac{\mathrm{d}I_L}{\mathrm{d}t} = I_R R'$$

Applying nodal analysis,

$$I_C + I_L + I_R = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}^2V}{\mathrm{d}t^2} + \frac{1}{R'C}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{V}{LC} = 0$$

We now have the corresponding damping coefficient and (undamped) resonant frequency

$$\gamma = \frac{1}{2R'C} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

as well as the quality factor

$$Q = \frac{\omega_0}{2\gamma} = R'\sqrt{\frac{C}{L}}$$

Then the corresponding conditions on the solution are as follows:

1. if $\gamma < \omega_0$, i.e. $Q > \frac{1}{2}$, then the circuit response is underdamped,
2. if $\gamma = \omega_0$, i.e. $Q = \frac{1}{2}$, then the circuit response is critically damped,
3. if $\gamma > \omega_0$, i.e. $Q < \frac{1}{2}$ then the circuit response is overdamped.

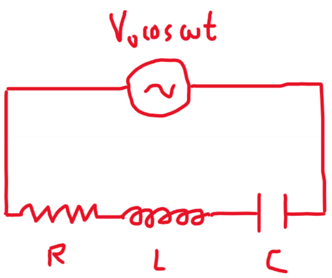
For the quality factors of the series RLC circuit and the parallel R'LC circuit to be the same,

$$\frac{1}{R}\sqrt{\frac{L}{C}} = R'\sqrt{\frac{C}{L}} \quad \Longrightarrow \quad RR' = \frac{L}{C}.$$

1.4 AC Circuits

Now, we shall investigate circuits with sinusoidal time dependence: **alternating current** (AC) circuits. Such circuits usually contain AC voltage or current sources, which exhibit sinusoidal voltages or currents as opposed to constant voltages or currents. These circuits are typically easier to analyse compared to DC RLC circuits, using a useful trick to analyse these circuits that we shall see.

Let's motivate our analysis by first starting with an example containing an AC voltage source. Consider a circuit with a resistor R, capacitor C and inductor L in series with an an AC voltage source $V = V_0 \cos\omega t$.



Solving for the circuit response gives

$$V_\mathrm{C} + V_\mathrm{L} + V_\mathrm{R} = V \quad \Longrightarrow \quad \frac{\mathrm{d}^2V_\mathrm{C}}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}V_\mathrm{C}}{\mathrm{d}t} + \frac{V_\mathrm{C}}{LC} = \frac{V_0}{LC}\cos\omega t$$

This linear differential equation has a solution that consists of two contributions: a **transient** response and a **steady state** response. The transient response corresponds to the *homogeneous* solution to this differential equation – we've already derived this in the previous section.

In AC circuits, we're only interested in the steady state response, corresponding to the *particular* solution of the differential equation. Usually, when we solve for the particular solution in this case, we would guess for a solution of the form

$$V_{\mathrm{p}} = A\cos\omega t + B\sin\omega t$$

and solve for the coefficients using the differential equation. However, when dealing with AC circuits, typically we prefer to work with *complex quantities*, treating sinusoidal functions as the real part of some complex number, a **phasor**, multiplied by a complex exponential! It turns out that phasors can not only just represent sinusoidal functions – they can also simplify our analysis of RLC circuits greatly, *completely eliminating our need to solve any differential equations.*

Remark. In this entire section, we shall denote the imaginary unit as $j = \sqrt{-1}$, rather than i. This is usually a notation that engineers adopt rather than physicists, but it is more convenient for us to adopt this notation in this chapter to prevent ambiguity with currents that could be denoted as i.

1.4.1 Phasors

A sinusoidal varying quantity A with frequency ω, which we say is in the *time domain,* can be represented as a phasor $\tilde{A}$, which we say is in the *frequency domain*, as follows:

$$A(t) = \mathrm{Re}\left(\tilde{A}e^{j\omega t}\right) \quad\Longleftrightarrow\quad A \leftrightarrow \tilde{A} \tag{13}$$

A phasor is a complex number that, when expressed in polar form, has a magnitude A_0 and phase φ, which allows us to write

$$\mathrm{Re}\left(\tilde{A}e^{j\omega t}\right) = \mathrm{Re}\left(A_0 e^{j(\omega t+\varphi)}\right) = A_0\cos(\omega t+\varphi)$$

Therefore, this gives us a way to convenient way to represent sinusoidal functions directly as phasors,

$$A_0\cos\left(\omega t+\varphi\right) \leftrightarrow A_0 e^{j\varphi} \tag{14}$$

It is also clear that the usual operations of addition/subtraction of phasors and multiplication/division of phasors with complex numbers also do the same operations on their corresponding sinusoidal functions, so we can convert any sinusoidal dependents from the time domain into the frequency domain, freely carry out operations in the frequency domain and convert the result back into the time domain.

The benefit of this notation comes to light when we consider taking the time derivative of quantities:

$$\frac{\mathrm{d}A}{\mathrm{d}t}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\left(\tilde{A}e^{jwt}\right) = \mathrm{Re}\left(j\omega\tilde{A}e^{jwt}\right)$$

Then we can represent the derivative of a sinusoidal quantity, which is another sinusoidal quantity, as

$$\frac{\mathrm{d}A}{\mathrm{d}t} \leftrightarrow j\omega\tilde{A} \tag{15}$$

This turns out to be extremely useful when dealing with circuits!

1.4.2 Impedance and Ohm's Law

When dealing with RLC components in AC circuits, we can use phasors to greatly simplify their associated equations. The respective equations expressed in phasor quantities are:

$$V = IR \quad \leftrightarrow \quad \tilde{V} = \tilde{I}R \tag{16}$$

$$I = C\frac{\mathrm{d}V}{\mathrm{d}t} \quad \leftrightarrow \quad \tilde{V} = \frac{\tilde{I}}{j\omega C} \tag{17}$$

$$V = L\frac{\mathrm{d}I}{\mathrm{d}t} \quad \leftrightarrow \quad \tilde{V} = j\omega L\tilde{I} \tag{18}$$

From these equations, we can conclude that in the frequency domain, **all RLC components behave as resistors**! This is because the voltage and current are directly proportional in the frequency domain, so we can apply Ohm's law even for capacitors and inductors. Here, the analogous resistances are referred to as the **impedance** instead, which is also a complex number (but not a phasor). Then the impedances of RLC components are

$$Z_{\mathrm{R}} = R \qquad Z_{\mathrm{C}} = \frac{1}{j\omega C} \qquad Z_{\mathrm{L}} = j\omega L \tag{19}$$

This principle allows us to treat AC circuits as DC circuits with complex resistances, i.e. impedances, and completely eliminates our need to solve differential equations.

To illustrate their use, we go back to the example in the beginning of this section: a circuit with resistor R, capacitor C and inductor L in series with an AC voltage source $V = V_0 \cos\omega t$. Let's suppose we want to solve for the **peak current** I_{max}. To do so, we shall first represent the voltage as a phasor:

$$V = V_0 \cos\omega t \quad \leftrightarrow \quad \tilde{V} = V_0$$

Here, the phasor is completely real since the phase is zero. Then, treating the RLC components as resistors in series, the effective impedance of the three components is

$$Z_{\mathrm{eff}} = Z_R + Z_C + Z_L = R + \frac{1}{j\omega C} + j\omega L = R + \left(\omega L - \frac{1}{\omega C}\right) j$$

We can then solve for the current in the frequency domain using Ohm's law,

$$\tilde{I} = \frac{\tilde{V}}{Z_{\mathrm{eff}}} = \frac{V_0}{R + \left(\omega L - \frac{1}{\omega C}\right) j}$$

Finally, the peak current is the magnitude of the phasor, so we have

$$I_{\mathrm{max}} = \left| \frac{V_0}{R + \left(\omega L - \frac{1}{\omega C}\right) j} \right| = \frac{|V_0|}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

With that, we've solved the problem without the use of a single derivative! This illustrates the power of phasors when solving AC circuits. Everything that works for DC circuits – Kirchhoff's laws, mesh and nodal analysis, etc. all work for AC circuits in the frequency domain!

1.4.3 Root Mean Square Quantities

We're almost ready to conclude our discussion AC circuits, as we're already equipped with the techniques required to solve AC circuits with our knowledge of DC circuits. However, there are a few more concepts we need to cover before we can finish. It is important to discuss one quantity that is often used in the context of AC circuits.

The **root mean square** value of a function f that has period T is defined as

$$f_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T (f(t))^2 \, \mathrm{d}t} \tag{20}$$

One can see that this is analogous to the mean,

$$\overline{f} = \frac{1}{T}\int_0^T f(t)\, \mathrm{d}t$$

but typically we're uninterested in the means of voltages and currents in AC circuits, since they're zero.

The simple evaluation of an integral shows that for pure sinusoidal voltages and currents, the root mean square value is

$$V_{\text{rms}} = \frac{V_{\text{max}}}{\sqrt{2}} \qquad I_{\text{rms}} = \frac{I_{\text{max}}}{\sqrt{2}} \tag{21}$$

Root mean square values are used more often than the peak values, since they're the "effective value" that a voltmeter or ammeter would measure. When a value is given for an AC component and it is not specified whether it refers to the peak or root mean squared value, it is typically the root mean squared value (but it does not hurt to clarify when in doubt), so it is important to remember to convert any root mean squared values into their peak values and back when necessary.

1.4.4 Instantaneous and Average Power

One more subtle thing to cover is dealing with *power*. Let's suppose that the voltage and current through some element is

$$\tilde{V} = V_0 e^{j\varphi} \qquad \tilde{I} = I_0 e^{j\psi}$$

Our trick using phasors works for voltage and current works precisely because all operations dealing with them are linear. However, power is defined as $P = VI$, meaning it is no longer a linear quantity. Hence, it is **incorrect** to say the instantaneous power is $\text{Re}\left(\tilde{V}\tilde{I}e^{j\omega t}\right)$. Rather, we have

$$P(t) = \text{Re}\left(\tilde{V}e^{j\omega t}\right)\text{Re}\left(\tilde{I}e^{j\omega t}\right) = V_0 I_0 \cos\left(\omega t + \varphi\right)\cos\left(\omega t + \psi\right) \tag{22}$$

Sometimes, we're also interested in the *average power*, which is the mean power (as opposed to the root mean squared power, which we do not care about). Evaluating another simple integral also lets us conclude that the average power

$$\overline{P} = \frac{1}{2}V_0 I_0 \cos\left(\varphi - \psi\right) \tag{23}$$

If we only want to deal with phasors, we can express this in terms of phasors by converting the cosine term into the real part of a complex exponential:

$$\overline{P} = \text{Re}\left(\frac{1}{2}V_0 I_0 e^{j(\varphi-\psi)}\right) = \text{Re}\left(\frac{1}{2}\tilde{V}\tilde{I}^*\right) \tag{24}$$

where z^* is the complex conjugate of z. Using these definitions, we can also show that the average power absorbed by individual RLC components are

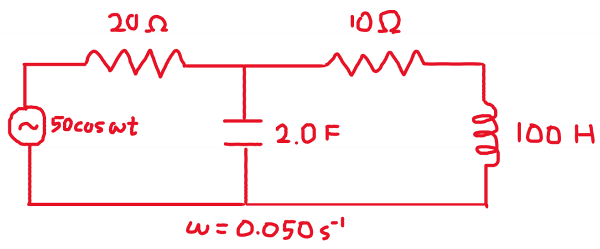
$$\overline{P}_{\text{R}} = \frac{1}{2}I_{\text{max}}^2 R = I_{\text{rms}}^2 R = \frac{V_{\text{max}}^2}{2R} = \frac{V_{\text{rms}}^2}{R} \qquad \overline{P}_{\text{C}} = \overline{P}_{\text{L}} = 0 \tag{25}$$

This asset of formulas will come useful when it comes to analysing power delivered or absorbed in a circuit.

1.4.5 AC RLC Circuits

Finally, we can put everything we've learned so far together to solve an AC RLC circuit question.

Example 1.6. Find the average power absorbed by each of the circuit elements below and verify that they sum to zero.



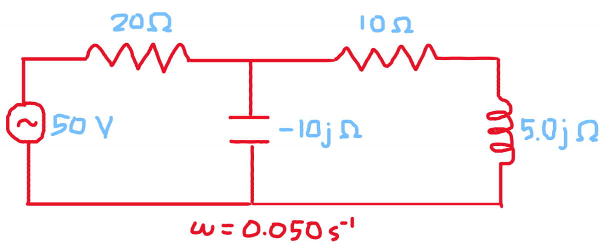
In this AC circuit, $\omega = 0.050\ \text{s}^{-1}$. Let us first convert everything to the frequency domain. The AC voltage source as a phasor is

$$\tilde{V} = 50\ \text{V}$$

The impedances of the resistors are just their resistances, and the impedances of the capacitor and inductor are

$$Z_\text{C} = \frac{1}{j\omega C} = -10j\ \Omega \qquad Z_\text{L} = j\omega L = 5j\ \Omega$$

Then our equivalent circuit in frequency domain is shown below.



We already know that the capacitor and inductor deliver zero net average power. We just need to calculate the currents through the resistors and the voltage source. Let the left and right loop currents be $\tilde{i}_1$ and $\tilde{i}_2$ respectively. Applying mesh analysis,

$$-50 + (20 - 10j)\tilde{i}_1 + (10j)\tilde{i}_2 = 0 \qquad (10j)\tilde{i}_1 + (10 - 5j)\tilde{i}_2 = 0$$

Solving for the loop currents, we get

$$\tilde{i}_1 = (1.7073 + 0.3659j)\ \text{A} \qquad \tilde{i}_2 = (0.9756 - 1.2195j)\ \text{A}$$

and we can get their peak values by taking their magnitudes,

$$i_{1,\text{max}} = 1.746\ \text{A} \qquad i_{2,\text{max}} = 1.561\ \text{A}$$

Finally, we can calculate the average power absorbed by each component:

$$\overline{P}_{50\ \text{V}} = \text{Re}\left(\frac{1}{2}\tilde{V}\tilde{i}_1^*\right) = -42\ \text{W}$$

$$\overline{P}_{20\ \Omega} = \frac{1}{2}i_{1,\text{max}}^2 R_{20\ \Omega} = 30\ \text{W}$$

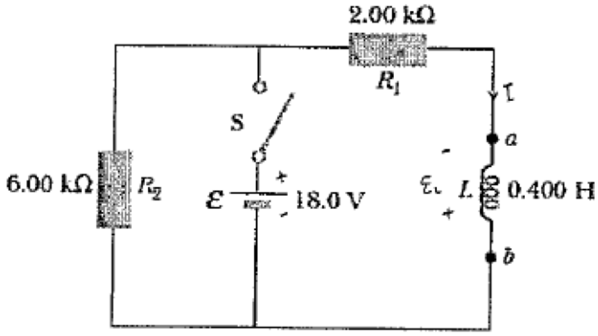
$$\overline{P}_{10\ \Omega} = \frac{1}{2}i_{2,\text{max}}^2 R_{10\ \Omega} = 12\ \text{W}$$

and we can clearly see that the sum of these values are zero, i.e. energy is conserved.

2 Problems

Problems are arranged in roughly increasing difficulty.

Problem 2.1 (SPhO 2013)**.** In the figure below, the switch is closed for $t < 0$, and steady-state conditions are established. The switch is opened at $t = 0$. (i) Find the initial voltage $\varepsilon_{\mathrm{L},0}$ across L just after $t = 0$. Which end of the coil is at higher potential, a or b? (ii) Sketch graphs of the currents in R_1 and R_2 as a function of time, treating the steady-state directions as positive. Show the values before and after $t = 0$. (iii) How long after $t = 0$ would I across R_2 have the value 2.00 mA?

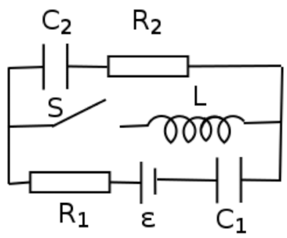


Problem 2.2 (SPhO 2006)**.** A bicycle dynamo consists of a small permanent magnet which is fixed at its center to the axle of one of the bicycle wheels, which has a radius r. Flux from the magnet links (via an iron yoke) a coil of self-inductance L, the coil being connected to a lamp of resistance R. The flux Φ linking the coil can be approximated as a sinusoidally-varying quantity of the form $\Phi(t) = \Phi_0 \cos(\omega t)$. (i) Establish the equations satisfied by I_0 and ε, if the current in the circuit is $I(t) = I_0 \cos(\omega t + \varepsilon)$, and show that $\tan\varepsilon = \frac{Rr}{vL}$, where v is the road speed of the bicycle. (ii) Find how the power P delivered to the lamp varies with v, and show that $P \leq \frac{R\Phi_0^2}{2L^2}$.

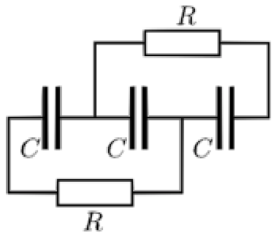
Problem 2.3. (Ricardo) Determine the charges stored by the capacitors if the switch is closed for a long time, given that the capacitors start from a configuration with zero stored charge. What if the switch is opened from the start instead?

(a) Switch closed for a long time: find the final charges on the capacitors.

(b) Switch opened from the start: find the final charges on the capacitors.

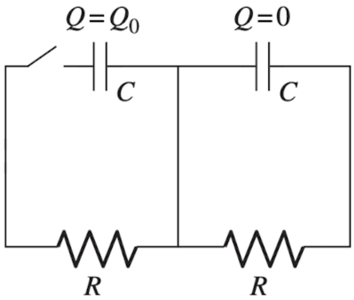


Problem 2.4. Three identical capacitors are placed in series and charged with a battery of emf $\mathcal{E}$. Once they are fully charged, the battery is removed, and simultaneously two resistors are connected as shown.

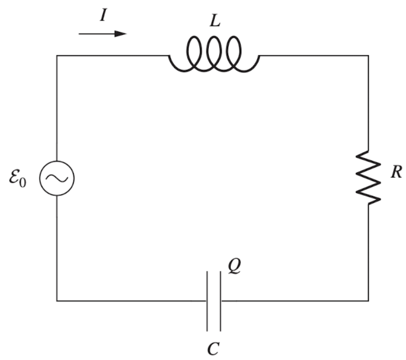


Find the heat dissipated on each of the resistors after a long time.

Problem 2.5. (Purcell) The circuit below contains two identical capacitors and two identical resistors. Initially, the left capacitor has charge Q_0 (with the left plate positive), and the right capacitor is uncharged. If the switch is closed at $t = 0$, find the charges on the capacitors as functions of time.



Problem 2.6. This problem illustrates how phasor diagrams can be useful. Consider the series RLC circuit below:



(a) Write KVL for circuit; given that I takes the form $I(t) = I_0 \cos(\omega t + \phi)$, show that the KVL equation can be written as

$$\omega L I_0 \cos(\omega t + \phi + \pi/2) + R I_0 \cos(\omega t + \phi) + \frac{I_0}{\omega C} \cos(\omega t + \phi - \pi/2) = \mathcal{E}_0 \cos \omega t \qquad (26)$$

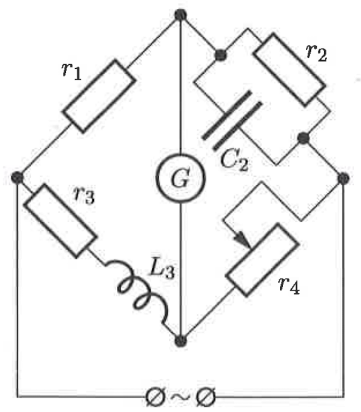
(b) At any given time, the four terms in the equation above can be considered to be the real parts of four vectors in the complex plane. Draw a quadrilateral that represents the fact that the sum of the three terms on the left side of the equation equals the term on the right side.

(c) Use your quadrilateral to determine the amplitude I_0 and phase ϕ of the current.

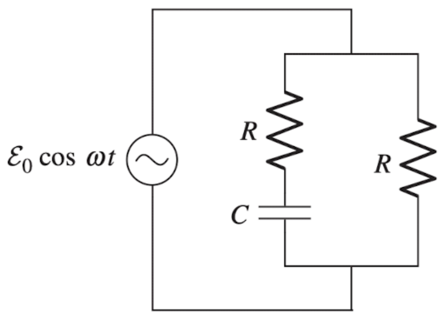
Problem 2.7. USAPhO 2011, Problem B1.

Problem 2.8. (RPhO 2009) The Wheatstone bridge circuit is used to determine a capacitance and a leakage resistance (C_2, r_2) of a capacitor. The bridge is balanced when a harmonic alternating voltage is applied, and the balance persists even under variations of the voltage frequency. Determine C_2 and r_2 given

$$r_1 = 2500\ \Omega, \qquad r_3 = 10\ \Omega, \qquad L_3 = 1\ \text{H}, \qquad r_4 = 800\ \Omega.$$



Problem 2.9. The circuit in below has two equal resistors R and a capacitor C. The frequency of the emf source, $\mathcal{E}_0 \cos \omega t$, is chosen to be $\omega = \dfrac{1}{RC}$.

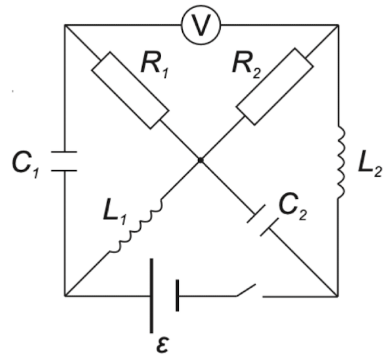


(a) What is the total complex impedance of the circuit? Give it in terms of R only.

(b) If the total current through the circuit is written as $I_0 \cos(\omega t + \phi)$, what are I_0 and ϕ?

(c) What is the average power dissipated in the circuit?

Problem 2.10. For the circuit shown in the figure, $R_1 = 3R$, $R_2 = R$, $C_1 = C_2 = C$ and $L_1 = L_2 = L$. The e.m.f. of the battery is E. Initially the switch is closed and the system is operating in a stationary regime.

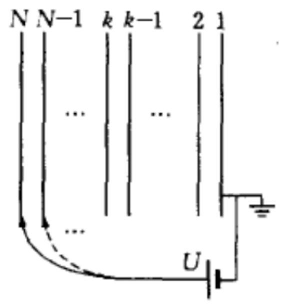


(a) Find the reading of the voltmeter in the stationary regime.

(b) Now, the switch is opened. Find the reading of the voltmeter immediately after the opening.

(c) Find the total amount of heat which will be dissipated on each of the resistors after opening the switch, and until a new equilibrium state is achieved.

Problem 2.11. N uncharged capacitors are placed equal distances apart, and numbered as shown. The area of the capacitors is much larger than the distances between them. Initially, capacitor 1 is connected to ground, along with a source of voltage U. Then, a lead connected to the source touches capacitor N, then $N-1$, etc. until capacitor 2.

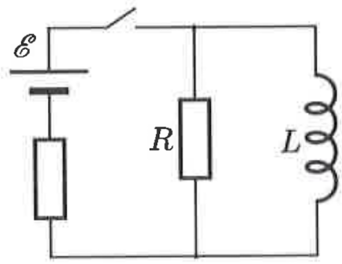


(a) Find the ratio of the final charges on the k-th capacitor to the 1st capacitor, $|q_k| : |q_1|$.

(b) Find the electric potential on the k-th capacitor, U_k.

3 Advanced Problems

Problem 3.1. (RPhO Final Stage) Parameters $\mathcal{E}, R, L$ of the circuit shown in the figure are known. Initially the switch is open and there is no current in the circuit containing the inductor. Then the switch was closed for some time and opened again. It is known that a charge q_0 passed through the inductor when the switch was closed. The net heat released in the circuit after the switch was opened is Q_0.



Assuming the circuit elements to be ideal, determine:

(a) the current I_0 flowing through the inductor just before the switch was opened;

(b) the charge q_1 passed through the resistor R when the switch was closed;

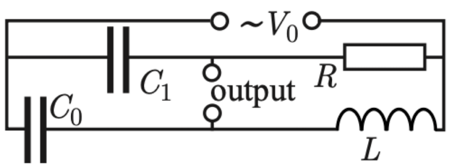
(c) the charge q_2 passed through the resistor R after the switch was opened;

(d) the net work A done by the DC power source during the whole process;

(e) the net heat Q released in the circuit when the switch was closed.

Hint: determine a relation between a charge passed through the resistor R and a change of magnetic flux in the inductor.

Problem 3.2. An alternating voltage with amplitude V_0 and angular frequency ω_0 is applied to this circuit.



(a) For which angular frequency ω_0 would the output voltage be infinite?

(b) Now let $\omega = 2\omega_0$. The capacitance C_1 is chosen such that the phase difference ϕ between the input and output voltages is maximum (parameters C_0, L and R are not changed). Find the phase shift ϕ and the output voltage amplitude V_{out}.